This technical note presents a numerical corrective technique that allows control of nonlinearity in a mass-spring system (MSS) independent of its spring constants or system topology. The governing equations of MSS in the form of ordinary differential equations or a regular function accompanied by any boundary or initial condition as known constraints, are employed to modify the results. A least-squares algorithm coupled with the finite difference method is used to discretize the basic residual function implemented in this corrective technique. This numerical solution is applicable to both static and dynamic MSS. This technique is easy to implement and has accuracy similar to that of the equivalent finite element method (FEM) solution to the same system whereas solutions are obtained in a fraction of the CPU time. The proposed technique can also be used to smooth solutions from other methods such as FEM or boundary element method (BEM).

**Keywords:** mass-spring system, finite difference method, least square algorithm, finite element method, soft tissue mechanics

Mass-spring systems (MSS) have been extensively used in computational simulation of soft tissues mechanics over the past 15 years. They are also used in modeling of deformable objects for facial animation, animation of artificial animals, cloth modeling, and recently in surgical simulation (Miller, 1988; Terzopoulos et al., 1987; Lee et al., 1995; Platt & Badler, 1981; Tu & Terzopoulos, 1994; Breen et al., 2001; Eberhardt et al., 1996; Carignan et al., 1992; Choi & Ko, 2002; Vassiley et al., 2001; Volino et al., 1995; Brown et al., 2001). Although, MSS are expeditious, they suffer from the lack of accuracy as compared with other numerical techniques such as the finite element method (FEM) (Provot, 1995). To address this shortcoming, numerical corrective techniques have been developed for the MSS. The approaches used include the inverse dynamics procedure to eliminate super elongation of the springs, the implicit integration method to take large time steps, and the heuristic method of handling post-buckling instability for robust modeling (Gelder et al., 1998; Choi & Ko, 2002, Luciani et al., 1991; Bhat et al., 2003).

Both isotropic and anisotropic elastic materials can be found among the objects to animate. For instance, most soft tissues are strongly anisotropic owing to their fiber structure and/or composite nature. One of the main limitations with the mass-spring systems is that neither isotropic nor anisotropic materials can be generated and controlled easily (Gelder, 1998; Jojic & Huang, 1997; Maciel et al., 2003; Terzopoulos et al., 1989).

In this study, a new numerical corrective technique for use with MSS is developed by which the accuracy of MSS results is enhanced by utilizing the material properties and the boundary and/or initial conditions of the system. FORTRAN custom code is used to implement the corrected MSS technique in this study. Application of this new corrective technique is illustrated by three examples. These include the force displacement of a nonlinear spring; a highly nonlinear displacement of a nonlinear spring; and a highly nonlinear stress-strain relationship of a polymer hydrogel, which is a biomaterial that mimics cardiovascular tissue properties (Millon et al., 2006).
Method

Governing Equation for MSS

The internal force of a MSS that is due to the changes of spring length and velocity takes the form of

\[ \vec{F}_{\text{int}} = [K(\cdot)|x - x_0| + D, \frac{x \cdot \vec{v}}{|x|}, \frac{x}{|x|}] \]  

(1)

where \( \vec{x} \) and \( \vec{v} \) are position and velocity vectors between the two ends of the springs and \( |x| \) and \( x_0 \) are the current and initial length of the spring, respectively. The terms \( K \) and \( D \) are linear constants of the MSS elements representing the spring and dashpot constants.

Equation 1 can be represented in the following nonlinear form:

\[ \vec{F}_{\text{int}} = K(\cdot) \cdot x_k + D(\cdot) \cdot x_D \]  

(2)

\[ x_k = (|x| - x_0) \cdot \frac{\vec{x}}{|x|} \]  

(3)

\[ x_D = \frac{\vec{x} \cdot \vec{v}}{|x|} \cdot \frac{\vec{x}}{|x|} \]  

(4)

Equations 3 and 4 are defined as directional vectors, and the nonlinear functions \( K(\cdot) \) and \( D(\cdot) \) represent spring stiffness and damping functions. Newton’s equations of motion can be rearranged in the following form:

\[ \vec{F}_{\text{int}}(k + 1) = \frac{M}{(\Delta t)^2}[\Delta x(k) - \Delta x(k - 1)] - F_{\text{ext}} \]  

(5)

where \( \Delta x(k) \) and \( \Delta x(k - 1) \) are position changes and \( F_{\text{ext}} \) refers to the external force as boundary conditions. Given the input and output data, nonlinear functions \( K(\cdot) \) and \( D(\cdot) \) can be approximated using, for instance, ordinary differential equation (ODE) numerical solvers or neural network approaches.

Corrective Technique to MSS

MSS is governed by a set of ODEs as mentioned above. There is also additional information in the form of boundary or initial conditions that can also be formulated. This additional information, taken together with the governing equations, can be expressed as

\[ R(u_i, \zeta_j, \eta_k) = \sum_j \frac{1}{2}(\vec{u}_j - u_i)^2 + \sum_j \zeta_j F(u) + \sum_k \eta_k T(u) \]  

(6)

where \( R \) is the residual function to the MSS, \( j \) is the number of internal points of the domain to be corrected, and \( \zeta_j \) and \( \eta_k \) are the Lagrange multipliers. The certain quality to be corrected is displacement \( \vec{u}(x) \) available in \( i \) points. The corrected values of the \( \vec{u}(x) \) are considered as \( u_i(x) \). The governing equation of the MSS is \( F(u) = 0 \), where order of \( F \) is either 0 (static) and 2 (time dependent). Also, the additional information from the system such as boundary and initial conditions is \( T(u_i) = 0 \) and \( T \) is a known function of \( u \). Equation 6 is solved for \( u_i \) by minimizing the residual function \( R \).

It should be noted that \( i \) and \( j \) have different ranges: \( i \) refers to all points of the domain including boundary points and additional constraints, whereas \( j \) refers only to the internal points of the domain. The term \( k \) denotes the number of constraints where \( T(u_i) \) is defined. The residual function \( R \) should be minimized on the entire domain by taking the first derivative of the function \( R \) with respect to \( u \), as such:

\[ \frac{\partial R}{\partial u} = \sum_i (\vec{u} - u) + \sum_j \zeta_j \frac{\partial F}{\partial u} + \sum_k \eta_k \frac{\partial T}{\partial u} = 0 \]  

(7)

Also the first derivative of the function \( R \) with respect to \( \zeta_j \) and \( \eta_k \) delivers the following complementary set of equations:

\[ F(u) = 0 \]  

(8)

\[ T(u) = 0 \]  

(9)

The function \( F \) can be replaced by the equivalent finite difference forms calculated for the points \( j \). Equations 6 and 7 are now reduced to a set of algebraic equations, with \( u \) being the only variable. These sets of equations can be solved by any relevant numerical technique as they are in the form of linear or nonlinear algebraic equations. The order \( r \) of function \( F \) defines the number of connected points that are to be solved in each step. When the order of \( F \) is 2 (time dependent), three consecutive values, \( u_{i-1}, u_i, \), and \( u_{i+1} \) are defined and consequently \( j = i - 2 \) as \( j = i - r \), because Equations 8 and 9 can only be written for the internal points of the domain.

Equation 6 can also be written in matrix form, as such:

\[ R = \frac{1}{2}(\vec{u} - u)^2 + (F(u))^T \psi \]  

(10)

where

\[ \psi = \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \]  

(11)

and eventually the matrix form of the error function \( R \) takes the form of

\[
\begin{bmatrix}
1 & B^T \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\xi \\
\eta \\
0
\end{bmatrix} = 
\begin{bmatrix}
\vec{u} \\
0
\end{bmatrix}
\]  

(12)
where \( I \) is the unit diagonal matrix and \( B \) is the matrix resulting from the additional Equations 8 and 9, which is of the order \( i + j + k \). For ease of calculation, Equation 10 can be broken down to two lower systems of the order \( i \) and \( j + k \). The new partitioning of Equation 10 takes the matrix form, as such:

\[
Iu + B^T \psi = \bar{u} \tag{13}
\]

\[
Bu = 0 \tag{14}
\]

Equation 13 can be multiplied by \( B \):

\[
Bu + B^T \psi = B \bar{u} \tag{15}
\]

Using Equation 15, the Lagrange multipliers matrix can be computed as follows:

\[
\psi = SB \bar{u} \tag{16}
\]

and

\[
u = Y \bar{u} \tag{17}
\]

where \( Y \) is defined as

\[
Y = I - B^T SB \tag{18}
\]

which is considered the corrective matrix where matrix \( S \) is defined as

\[
S = (BB^T)^{-1} \tag{19}
\]

The order of matrix \( S \) is \((j + k) \times (j + k)\) and the correction matrix \( Y \) is essentially a square matrix of order \( i \times i \). The correction matrix is unique for a given MSS and once it is solved it can be retained as a modulus for the MSS. It can also be proven that the Lagrange multipliers \( \xi_j \) and \( \eta_k \) contribute to correction of the results, and, therefore, they can be considered as correction factors as well.

Now, the set of Equations 6 to 19 can appropriately interpret the MSS and can be solved for \( u \) and the multipliers \( \xi_j \) and \( \eta_k \). The solution is unique because it refers only to the corresponding set of governing equations and selected constraints of the MSS.

## Results

### Example 1—Time-Independent Application (A Nonlinear MSS)

#### Part I.

The force displacement of a nonlinear spring (\( f = ku^2 \), where \( f, u, \) and \( k \) refer to force, displacement, and the spring constant, respectively) under large deformation is considered. A linear spring is used for uncorrected results and is modified through the proposed technique. For this example, the governing equation is not a differential equation, but a second-order polynomial. The functions \( F \) and \( T \) take the form of

\[
F(u) = f - ku^2 = 0 \tag{20}
\]

where \( k \) is the spring constant, which is set to 1 for this example. The minimized residual function can be written as such:

\[
\frac{\partial R}{\partial u} = (\bar{u} - u) + 2ku \tag{22}
\]

where the \( B_{m,n} \) matrix is written as \((m = n = 2)\):

\[
B = \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix} \tag{23}
\]

Given \( B \) matrix (Eq. 23), Equation 13 is solved for the corrective matrix \( Y \). The \( Y \) matrix is now available for the uncorrected results. The exact solutions to this simple model for the linear and the used nonlinear spring constant are in the form of linear and parabolic functions, respectively. The uncorrected, exact, and modified results, accompanied by the Lagrange multipliers, are shown in Figure 1.

The trial function as uncorrected results for this simple system is linear. The corrected results are close to the result of the nonlinear function of Equation 20 with a deviation of less than 10%.

#### Part II.

The solution to Example 1 can be extended to a relatively complicated MSS. A rectangular polymeric plate \((5 \times 15 \text{ mm}^2)\) made of 10% polyvinyl alcohol (PVA) hydrogel is considered. The mechanical properties of the sample in the form of stress-strain relationship has been determined previously using a uniaxial tensile machine and are expressed as follows (Millon et al., 2006; Boukerrou et al., 2007):

\[
\sigma = -0.05923 + 0.0611e^{2.347e} \tag{24}
\]

where \( \sigma \) and \( \varepsilon \) are stress and strain, respectively, and Poisson’s ratio of the used PVA sample was assumed to be 0.5 (incompressibility).

The stress-strain behavior shows a highly nonlinear behavior similar to that of the porcine aortic root (Millon et al., 2006). An iterative modified Newton–Raphson method to MSS and a validated finite element solver (MSC/NASTRAN FE commercial software) for soft material considering material nonlinearity has been used. In this finite element solver, hyperelastic isotropic elements have been employed along with a classical Mooney–Rivlin soft material model. The model specifications have been listed in Table 1.

The sample was pulled up to 100% strain and the Cauchy stresses were calculated in each iteration at strains of 16%, 33%, 50%, 66%, 83%, and 100%. The FEM model and the deformation results at the mentioned strains has been shown in Figure 2.

The corrective technique to the results from MSS has also been calculated. The results of uncorrected MSS, modified MSS, and nonlinear FEM are shown in Figure 3.
Example 2—Time-Dependent Application. A vibrating spring with a linear spring constant is considered. The exact results, which are in the form of a sinusoidal function, are randomly manipulated by a factor of more than 20% to examine the robustness of the proposed numerical technique for a time-dependent example. Equations 8 and 9 take the form

\[ F(u) = \ddot{u} - ku = 0 \]  \hspace{1cm} (25)

\[ T(u) = 0 \]  \hspace{1cm} (26)

The modified MSS gives results that are close to FEM, with a percentage error of less than 5%. The corrected mass-spring approach is more accurate here than in the solution provided in Part 1 as the trial function in this example is already nonlinear. Also in this example, the accuracy approaches the existing finite element solution at 1/20 of the CPU time on a Pentium IV with a CPU speed of 2.4 GHz and 512 MB RAM.

The MSS and FEM models as uncorrected and precise models used in this study are shown in Figure 4.

**Figure 1**—Modeling of a parabolic spring with a linear spring constant in a MSS model—the exact solution and modified results are in the same range with 10% difference.

<table>
<thead>
<tr>
<th>Table 1 Analysis conditions of the FEM model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Size of plate</strong></td>
</tr>
<tr>
<td>Number of elements</td>
</tr>
<tr>
<td>Number of nodes</td>
</tr>
<tr>
<td>Material model</td>
</tr>
<tr>
<td>Hyperelastic model</td>
</tr>
<tr>
<td>Material constants</td>
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<tr>
<td>Model type</td>
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Figure 2 — The FEM model of a rectangular sample of PVA in a large deformation study. The gradient of deformation of the sample starting from original geometry (right) to the deformed geometry (left) corresponding to 0–100% deformation with a deformation increment of 16%, respectively.

Figure 3 — The corrected, uncorrected (MSS) and accurate (FEM) results for a 10% PVA rectangular plate ($5 \times 15 \text{ cm}^2$) in a deformation up to 100%.

Figure 4 — The FEM analysis including original and deformed models (A), the original MSS model (B), and the deformed MSS model (C).
where $\ddot{u}$ is the second derivative of the $u$ with respect to time and $k$ is the spring constant. The minimized residual function takes the form

$$\frac{\partial R}{\partial u} = (\ddot{u} - u) + \frac{\partial}{\partial u} (\ddot{u} - ku) = 0 \quad (27)$$

The expansion of the Equation 26 together with the implementation of finite difference to the Equation 25 gives the following set of equations:

$$(\pi_0 - u_0) + \ddot{\xi}_1 = 0 \quad (28)$$

$$(\pi_i - u_i) + (-2 + \hat{k})\ddot{\xi}_i + \ddot{\xi}_2 = 0 \quad (29)$$

$$(\pi_j - u_j) + \ddot{\xi}_{j-1} + (-2 + \hat{k})\ddot{\xi}_j + \ddot{\xi}_{j+1} = 0 \quad (30)$$

$$(u_n - u_n) + \ddot{\xi}_{n-1} = 0 \quad (31)$$

where $\ddot{\xi}_i = \ddot{\xi}_i (\pi^i (t)^2)$ and $\hat{k} = k_j (\pi^j (t)^2)$ and the $B_{m,n}$ matrix for this case takes the form of $(m = 6, n = 4)$:

$$B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 + k & 1 & 0 & 0 \\
1 & -2 + k & 1 & 0 \\
0 & 1 & -2 + k & 1 \\
0 & 0 & 1 & -2 + k \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \quad (32)$$

The $Y$ matrix is now calculated for the current MSS. The results of this solution and the exact solution for $k = 1$ are presented in Figure 5 (please see p. 277), including the Lagrange multipliers.

Although the manipulated data are designed to hold more than 20% error, the discrepancy between corrected and exact results is smaller.

The proposed technique can also be implemented on regressing other parameters of a MSS (for instance, for the spring constant used in Example 2). For this case, the residual function ($R$) is the same except a new variable ($k$) is added. An additional equation can be obtained where the residual function, $R$, is minimized with respect to $k$. This equation takes the form

$$\sum_i u \psi = 0 ,$$

which is nonlinear and can be solved iteratively using the Newton–Raphson algorithm or similar approaches. The calculated stiffness of the spring was $k = 1.2$ for uncorrected results.

**Discussion**

A corrective numerical technique to modify the MSS results independent of the system topology or spring constants has been developed. Illustrated examples of both static and dynamic systems indicate the successful implementation of this approach, demonstrating that corrections of more than 80% can be achieved. In the dynamic example of the vibrating spring, the corrective technique was able to improve the results to within 10% of the FEM results. In summary, the corrective numerical technique presented in this study is of general applicability to both static and dynamic application of MSS. Since the proposed technique is computationally efficient and easy to implement, it can also be employed to smooth solutions obtained using other numerical techniques such as finite element, boundary element, or finite difference methods to enhance the accuracy of the solutions.

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References


Figure 5 — Motion of the vibrating spring for k = 1, manipulated and modified results.


